derivative of W.

Th. Q No → Let E be a nls and M be a closed linear subspace of E, for any $[x] \in E/M$, if we define,

$$\|[x]\| = \inf.\{\|x+u\| : u \in M\}$$

then E/M becomes a nls moreover, if E is Banach space then E/M is also a ~~Ban~~ Banach space.

Or, Q No → Let M be a closed linear subspace of a normed linear space E. If the norm of a coset $[x] = \bar{x} + M$ in the quotient space E/M is defined by,

$$\|x + M\| = \inf.\{\|x+u\| : u \in M\}.$$

then E/M is a normed linear space. Further if E is a Banach space, then E/M is also a Banach space.

~~Proof~~ :- Clearly, $\|[x]\| \geq 0$ for every $[x] \in E/M$.
Also $\|[x]\| = 0 \Rightarrow \|x + M\| = 0$ i.e. $\inf\{\|x + v\| : v \in M\} = 0$
then, there exist a sequence $\{z_n\}$ in M such that $\lim_{n\to\infty} \|x + z_n\| = 0 \Rightarrow \|x + z_n\| = 0$ for every i.e. $z_n = -x$ for every n. i.e. $z_n \to -x$ as $n \to \infty$ Since M is closed.

$\therefore -x \in M \Rightarrow x \in M$ ($\because M$ is a linear subspace)

$\Rightarrow x + M = M$, the zero vector of E/M.

$\Rightarrow [x] = [0]$, So $\|[x]\| = 0$

$\Rightarrow [x] = 0$

Again, $[x] = 0 = M \Rightarrow x + M = M \Rightarrow \|x + M\| = \|M\| = 0$.

$\therefore \|[x]\| = 0$ iff $[x] = 0$

Now, for any scalar α. $\|[\alpha, x]\| = \|\alpha x + M\| = \inf\{\|\alpha x + x\| : v \in M\}$

$= \inf\{\|\alpha x + \alpha v\| : v \in M\}$

$= |\alpha| \inf\{\|x + v\| : v \in M\}$

$= |\alpha| . \|[x]\|$.

Now, for any $[x], [y] \in E/M$. $[x] + [y] = x + y + M$.
i.e. $(x + M) + (y + M) = (x + y) + M$.
Since $\|x + M\| = \inf\{\|x + v\| : v \in M\}$

$\therefore$ there exists a sequence $\{z_n\}$ in M such that

$\lim_{n\to\infty} \|x + z_n\| = \|x + M\|$

Similarly, there exist a sequence $\{w_n\}$ in M such that $\underset{n\to\infty}{Lt} \|y + w_n\| = \|y + M\|$.

Now, $\|x + y + M\| \leq \|x + z_n + y + w_n\|$

$\leq \|x + z_n\| + \|y + w_n\| \to \|x + M\| + \|y + M\|$ as $n \to \infty$.

$\therefore \|[x] + [y]\| \leq \|[x]\| + \|[y]\|$

Hence, E/M is a normed linear space.

Now, we show that E/M is a Banach space provided E is a Banach space. For this it is sufficient to show that every Cauchy sequence in E/M converges in E/M. Now, if we can find a subsequence of a Cauchy sequence in E/M converging to a point of E/M, then the Cauchy sequence it self converges to that point. We construct a subsequence $\{x_n + M\}$ the Cauchy sequence in E/M as follow:-

$\|(x_1 + M) - (x_2 + M)\| < \frac{1}{2}$

$\|(x_2 + M) - (x_3 + M)\| < \frac{1}{2^2}$

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$\|(x_n + M) - (x_{n+1} + M)\| < \frac{1}{2^n}$

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Let $y_1 \in x_1 + M$ be arbitrary we choose $y_2 \in x_2 + M$ Such that $\|y_1 - y_2\| < \frac{1}{2}$ we choose $y_3 \in x_3 + M$ such that $\|y_2 - y_3\| < \frac{1}{2^2}$. Continuing in this way we choose $y_n \in x_n + M$ such that,

$$\|y_{n-1} - y_n\| < \frac{1}{2^{n-1}}$$

Then $\{y_n\}$ is a sequence in E. Now for $m < n$,

$$\|y_m - y_n\| \leq \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \cdots$$
$$\cdots + \|y_{n-1} - y_n\| < \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}}$$

i.e. $$\|y_m - y_n\| < \frac{\frac{1}{2^m}\left(1 - \frac{1}{2^{n-m}}\right)}{1 - \frac{1}{2}} = \frac{1}{2^{m-1}}\left(1 - \frac{1}{2^{n-m}}\right)$$

$$< \frac{1}{2^{m-1}} \to 0 \text{ as } m, n \to \infty.$$

$\therefore \{y_n\}$ is Cauchy sequence in E. Since E is complete $y_n \to y \in E$.

Now, $\|(x_n + M) - (y + M)\| \leq \|y_n - y\| \to 0$ as $n \to \infty$

So, $(x_n + M) \to y + M$ as $n \to \infty$

$\therefore$ the Cauchy sequence in E/M converges to a point $y + M \in E/M$

Hence, E/M is a Banach space.

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Open mapping

Lemma:- If B & B' are Banach spaces, and if T is continuous linear transformation of B onto B', then the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B'.

Proof:- We denote by S_r & S_r' the open spheres of radius r centered on the origin in B & B'.

Now $x \in S_r \Leftrightarrow \|x\| < r \Leftrightarrow \left\|\frac{x}{r}\right\| < 1$.

$$\Leftrightarrow \frac{x}{r} \in S_1 \Leftrightarrow x \in r S_1.$$

and it can be easily seen that,

$$T(S_r) = T(r S_1) = r\,T(S_1).$$

Hence it suffices to prove that $T(S_1)$ contains some S_r'.

We begin by proving that $\overline{T(S_1)}$ contains some S_r'.

Now, $B = \bigcup_{n=1}^{\infty} S_n$ and since T is onto, $B' = T(B) = \bigcup_{n=1}^{\infty} T(S_n)$

Since B' is a complete metric space, it follows from Baire's category theorem that some $\overline{T(S_{n_0})}$ has an interior point y_0 which may be assumed to lie in $T(S_{n_0})$. The mapping $y \to y - y_0$ is a homeomorphism of B' onto itself,

So $\overline{T(S_{n_0})} - y_0$ has the origin as an interior point.

Now since $y_0 \in T(S_{n_0})$ $y_0 = Tx_0$, for some $x_0 \in S_{n_0}$

Now let $z \in T(S_{n_0}) - y_0$, Then $z = T(x) - y_0 = Tx - Tx_0$

(when $x \in S_{n_0}$) $= T(x - x_0)$, where $\|x - x_0\| \leq \|x\| + \|x_0\| < 2n_0$.

Hence, $x - x_0 \in S_{2n_0}$

Thus, $z \in T(S_{2n_0})$

Hence, $T(S_{n_0}) - y_0 \subseteq T(S_{2n_0})$

Therefore, $\overline{T(S_{n_0})} - y_0 = \overline{T(S_{n_0}) - y_0} \subseteq \overline{T(S_{2n_0})}$

Which shows that the origin is an interior Point of $\overline{T(S_{2n_0})}$. Since multiplication by any non-zero scalar is a homeomorphism of B' onto itself, so $\overline{T(S_{2n_0})} = \overline{2n_0 T(S_1)}$ $= 2n_0 \overline{T(S_1)}$ and it follows from this that the origin is an interior Point of $\overline{T(S_1)}$

So $S'_\epsilon \subseteq \overline{T(S_1)}$ for some +ve number ϵ.

We ~~conc~~ conclude the Proof by showing that $S'_\epsilon \subseteq T(S_3)$, which is clearly equivalent to $S'_{\epsilon/3} \subseteq T(S_1)$.

Let y be a vector in B' such that $\|y\| < \epsilon$.

Since, $y \in \overline{T(S_1)}$, there exists a vector x_1 in B such that $\|x_1\| < 1$ and $\|y - y_1\| < \epsilon/2$, where $y_1 = Tx_1$.

We next observe that $S'_{\epsilon/2} \subseteq \overline{T(S_{1/2})}$ so there exist x_2 in B such that $\|x_2\| < 1/2$ and $\|(y - y_1) - y_2\| < \epsilon/4$ Where $y_2 = T(x_2)$.

Continuing in this way we obtain a sequence (x_n) in B such that

$$\|x_n\| < \frac{1}{2^{n-1}} \text{ and } \|y - (y_1 + y_2 + \dots + y_n)\| < \frac{\epsilon}{2^n},$$

Where $y_n = T(x_n)$.

If we put $S_n = x_1 + x_2 + \cdots x_n$, then it follows from $\|x_n\| < \frac{1}{2^{n-1}}$, that (S_n) is a Cauchy sequence in B for which

$$\|S_n\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_n\|$$

$$< 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} < 2.$$

Since B is Complete,

So there exists a vector x in B such that $S_n \to x$ and $\|x\| = \|Lt.\, S_n\| = Lt.\, \|S_n\| \leq 2 < 3$, hence x is in S_3. Also the Continuity of T yields

$$T(x) = T(Lt.\, S_n) = Lt.\, T(S_n)$$

$$= Lt.\, (T(x_1) + \cdots + T(x_n))$$

$$= Lt.\, (y_1 + \cdots + y_n) = y.$$

Thus, $y = T(x)$ where $x \in S_3$.

Hence, $y \in T(S_3)$. Thus $S'_\epsilon \subseteq T(S_3)$.

Hence, $S'_{\epsilon/3} \subseteq T(S_1)$.

S. Patel M.U 92, 94, 96